A CHARACTERIZATION OF SQUARE BANACH SPACES†

BY

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ABSTRACT

Square Banach spaces are characterized among real Banach spaces in terms of the Alfsen-Effros structure topology on the extreme points of the dual ball. As a corollary, one has that the class of separable square spaces coincides with the class of separable G-spaces. It is also shown that for a G-space (hence for a square space) regularity of the quotient structure topology is equivalent to complete regularity, and that square spaces exist for which this topology is not regular.

Among the various classes of Lindenstrauss spaces which have been studied are the class of square spaces, introduced by Cunningham in [4], and the class of G-spaces, introduced by Grothendieck in [8]. It was shown in [4] that every square space is a G-space, and that non-separable G-spaces exist which are not square. (See, however, the remark following Corollary 2 below.) In [1] Alfsen and Effros introduced a structure topology on the set E of extreme points of the closed unit ball in the dual of a real Banach space. Denoting by E_σ the quotient space obtained by identifying antipodal points in E, P. Taylor [10] has proved that a Lindenstrauss space is a G-space if and only if the structure topology on E_σ is Hausdorff. The first result of this paper (Theorem 1) is the following analog for square spaces: a necessary and sufficient condition for a real Banach space X to be square is that the bounded structurally continuous functions on E_σ separate points in E_σ. (Note that this does not assume that X is a Lindenstrauss space.) As corollaries to this theorem we have:

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(i) Every separable G-space is square.

(ii) A separable real Banach space is square if and only if E_σ is structurally T_1 and normal.

(iii) The class of (Jerison) C_σ-spaces is contained in the class of square spaces.

Following these results we exhibit a base for the structure topology in the case of a G-space (Lemma 5) and use it to prove (Theorem 2) that for a G-space, structural regularity of E_σ implies complete regularity. Finally, we make use of the base in Lemma 5 to construct a square space for which E_σ is not structurally regular; this example shows that the inclusion in (iii) above is proper and that the hypothesis of separability cannot be omitted from (ii).

Only real Banach spaces will be considered, and we shall use the following notation and terminology. If X is a Banach space, then $E = \text{ext}\, B(X^*)$ denotes the set of extreme points of the closed unit ball $B(X^*)$ in the dual of X, and $\bar{E}$ is the weak* closure of E. Regarding X as a space of functions on E, let $\mathcal{M}(X)$ denote the algebra of multipliers of X; that is, a bounded real function f on E belongs to $\mathcal{M}(X)$ if for each $x \in X$ there is $y \in X$ satisfying $f(p)x(p) = y(p)$ for all $p \in E$. (This algebra was introduced in [3].) If T is a topological space, then $m(T)$ is the Banach space of all bounded real functions on T with the uniform norm, and $C(T)$ is the subspace of $m(T)$ consisting of continuous functions. If Ω is a compact Hausdorff space, then, as defined in [4], a closed subspace X of $m(\Omega)$ is an *upper semicontinuous* (USC) *scalar function space* (on Ω) provided

(i) $|x|$ is upper semicontinuous for each $x \in X$, and

(ii) X is invariant under multiplication by elements of $C(\Omega)$.

If X is an USC scalar function space on Ω, then for each $t \in \Omega$ the evaluation functional $\hat{t}$ is defined on X by $\hat{t}(x) = x(t)$ for all $x \in X$.

A square Banach space may be thought of as an USC scalar function space on some Ω, and E as the set of nonzero evaluation functionals and their negatives. These results from [4] and [5] are stated precisely in the following two lemmas.

LEMMA 1. ([4, p. 553]). *A Banach space is square if and only if it is isometric to an* USC *scalar function space.*

LEMMA 2 ([5, Th. 1]). *If X is an* USC *scalar function space on Ω, then*

$$E = \bigcup\{\{\hat{t}, -\hat{t}\} : t \in \Omega,\ \hat{t} \neq 0\}.$$

If X is a Banach space, the Alfsen-Effros structure topology on E is the topology whose nonempty closed sets are of the form $E \cap N$, where N is a nonzero weak* closed L-summand in X^* ([1, Prop. 3.3]). In case X is a Lindenstrauss space, the structure topology on E coincides with the *bifacial* topology on E introduced in [6]. (See [1, pp. 168, 169].) To prove the following lemma, one simply imitates the proof of [6, Prop. 4.8], using [1, Lem. 3.8].

LEMMA 3 (See [6, Prop. 4.8]). *If X is a Banach space, then for each $x \in X$ and $\alpha > 0$, the set*

$$\{p \in E : |x(p)| \geqq \alpha\}$$

is structurally compact.

In what follows, $E_\sigma = \{\{p, -p\} : p \in E\}$ has the quotient structure topology of E, and $C_s^b(E)$ (respectively, $C_s^b(E_\sigma)$) denotes the Banach algebra of all bounded, structurally continuous functions on E (respectively, E_σ) with the uniform norm.

LEMMA 4 ([1, Th. 4.9]). *If X is a Banach space, then $C_s^b(E) = \mathscr{M}(X)$.*

If X is a square space, then $\mathscr{M}(X)$ separates linearly independent points in E ([4, Lem. 7]). This fact (a short proof of which is included below) will be combined with Lemma 4 to prove part of the following theorem.

THEOREM 1. *A Banach space X is square if and only if $C_s^b(E_\sigma)$ separates points in E_σ.*

PROOF. Assume X is square. By Lemma 1 we may suppose that X is an USC scalar function space on a compact Hausdorff Ω. For each $p \in E$, let t_p be the (unique) point in Ω such that $p = \pm \hat{t}_p$. (See Lemma 2.) Let $\{p_1, -p_1\}$ and $\{p_2, -p_2\}$ be distinct points in E_σ. Then $t_{p_1} \neq t_{p_2}$ since $p_1 \neq \pm p_2$. Choose $f \in C(\Omega)$ with $f(t_{p_1}) \neq f(t_{p_2})$, and define g on E by $g(p) = f(t_p)$ for all $p \in E$. Then $g \in \mathscr{M}(X)$ since if $x \in X$, then $fx \in X$ and one easily verifies that $g(p)x(p) = fx(p)$ for all $p \in E$. Therefore $g \in C_s^b(E)$ by Lemma 4. Defining g' on E_σ by $g'\{p, -p\} = g(p)$ (the functions in $\mathscr{M}(X)$ are even), we have that $g' \in C_s^b(E_\sigma)$ and

$$g'\{p_1, -p_1\} \neq g'\{p_2, -p_2\}.$$

Conversely, suppose X is a Banach space with the property that $C_s^b(E_\sigma)$ separates points in E_σ. Let Ω denote the spectrum of $C_s^b(E_\sigma)$. To show that X is square, it is sufficient, by Lemma 1, to show that X is isometric to an USC scalar function space on Ω. Let $e: E_\sigma \to \Omega$ be the evaluation map: $e(t)(f) = f(t)$ for $t \in E_\sigma$ and $f \in C_s^b(E_\sigma)$. Then e is continuous and is injective because $C_s^b(E_\sigma)$ separates points in E_σ. Using the Axiom of Choice, from each member t of E_σ choose a point of E, say p_t. Then for each $x \in X$ define the function x' on Ω as follows. For $\omega \in \Omega$, let

$$x'(\omega) = \begin{cases} x(p_t) & \text{if } \omega = e(t) \\ 0 & \text{if } \omega \notin e(E_\sigma). \end{cases}$$

Then $x \to x'$ is an isometry of X into $m(\Omega)$ by the Krein-Milman theorem. Let $X' = \{x' : x \in X\}$. To see that X' is an USC scalar function space, let $x \in X$, let $\alpha > 0$, and denote by P the quotient map $P(p) = \{p, -p\}$ of E onto E_σ. Then

$$\begin{aligned}\{\omega \in \Omega : |x'(\omega)| \geqq \alpha\} &= \{\omega \in e(E_\sigma) : |x'(\omega)| \geqq \alpha\} \\ &= e(\{t \in E_\sigma : |x(p_t)| \geqq \alpha\}) \\ &= e \circ P(\{p \in E : |x(p)| \geqq \alpha\}),\end{aligned}$$

which is compact (hence closed) by Lemma 3 and the fact that $e \circ P$ is continuous. Thus $|x'|$ is upper semicontinuous. Let $f \in C(\Omega)$. Then $g = f \circ e \circ P$ is in $C_s^b(E)$. Hence, by Lemma 4, there exists $y \in X$ satisfying $g(p)x(p) = y(p)$ for all $p \in E$. One may routinely verify that $fx' = y'$ on Ω. Therefore X' is an USC scalar function space on Ω, and the proof of Theorem 1 is concluded.

Since the functions in $C_s^b(E)$ are even, we have the following equivalent formulation of Theorem 1.

COROLLARY 1. *A Banach space is square if and only if $C_s^b(E)$ separates linearly independent points in E.*

COROLLARY 2. *Every separable G-space is square.*

PROOF. In [6, p. 452] Effros applied [2, Chap. I, Sect. 10, Ex. 19] to show that if X is a separable G-space, then E_σ is Hausdorff. If instead, one applies [2, Chap. IX, Sect. 4, Ex. 15], one has that E_σ is also normal. Thus $C_s^b(E_\sigma)$ separates points in E_σ by Urysohn's lemma. The conclusion now follows from Theorem 1.

By [4, Th. 2] there are (non-separable) G-spaces which are not square. One of the lemmas to this theorem ([4, Lem. 5]) is incorrect when Y is separable. (This can be seen from the proof of Corollary 2.) However, as Cunningham has remarked, the lemma is correct with a revised proof when Y is non-separable.

The following result is obtained by combining Theorem 5, the proof of Corollary 2, and the fact that square spaces are G-spaces ([4, Th. 1]).

COROLLARY 3. *A separable Banach space is square if and only if E_σ is T and normal.*

In what follows, if A and B are sets, then $A - B$ denotes the complement of B in A.

COROLLARY 4. *Every C_σ-space is square.*

PROOF. If X is a C_σ-space, then $\bar{E}$ is contained in $E \cup \{0\}$, and E_σ is homeomorphic to E/R, where E has the relativized weak* topology and $pRq \Leftrightarrow p = \pm q$ ([7, Th. 9]). The quotient space E/R is locally compact Hausdorff since $E = \bar{E} - \{0\}$ is locally compact Hausdorff, the members of R are compact, and the saturation of each (relatively) closed subset of E is closed in E ([9, Th. 5.20, 3.12, 3.10]). Thus E/R is a Tychonoff space (T_1 and completely regular), hence the same is true of E_σ. Therefore $C_s^b(E_\sigma)$ separates points of E_σ, so X is square by Theorem 1.

If X is a square space for which E_σ is completely regular, then, in the proof of Theorem 1, the continuous injection e of E_σ into the spectrum Ω of $C_s^b(E_\sigma)$ is a homeomorphism of E_σ onto $e(E_\sigma)$, and Ω is the Stone-Čech compactification of E_σ. This is the case, for example, when X is separable or is a C_σ-space. (See Corollaries 3 and 4.) As Theorem 2 will show, this is also the case when E_σ is regular.

LEMMA 5. *If X is a G-space, then a base for the structure topology on E is the family $\{V_x: x \in X\}$, where $V_x = \{p \in E: x(p) \neq 0\}$.*

PROOF. Since $\bar{E} \subset [0,1]E$ ([10, Th. 1]), it follows from [6, Cor. 5.9] that for each x in X the set $E - V_x$ is structurally closed. To show that the sets V_x form a base, let U be structurally open and let $p \in U$. Then $E - U = N \cap E$, where N is a weak* closed subspace of X^*. Since $p \notin N$, there is $x \in X$ with $x(p) = 1$ and $x(N) = 0$. Then $p \in V_x \subset U$.

It follows from Lemma 5 that if X is a G-space, then a base for the (quotient) topology on E_σ is the family $\{P(V_x): x \in X\}$, where $P(p) = \{p, -p\}$ is the natural projection of E onto E_σ.

THEOREM 2. *If X is a G-space for which E_σ is regular, then E_σ is completely regular.*

PROOF. Since E_σ is regular by hypothesis, to prove that E_σ is completely regular it suffices to show that each point of E_σ has a completely regular neighborhood ([2, Chap. IX, Sect. 1, Ex. 14]). Let $t = P(p)$ be in E_σ and choose $x \in X$ with $x(p) \neq 0$. Then $P(V_x)$ is an open neighborhood of t, by Lemma 5. Further,

$P(V_x)$ is regular and is σ-compact by Lemma 3 since V_x is the union of the compact sets $\{q \in E: |x(q)| \geqq 1/n\}$, $n = 1,2,\cdots$. Therefore $P(V_x)$ is normal ([9, Prob. 5.Y and Lem. 4.1]), hence completely regular since points in E_σ are closed (because X is a Lindenstrauss space).

If X is a non-separable square space, then the structurally continuous functions on E_σ distinguish points in E_σ (Theorem 1) but, unlike the separable case, they may not distinguish points and closed sets. This is illustrated by the following example of a square space for which E_σ is not regular.

Let Ω denote the set of all ordinals less than or equal to the first uncountable ordinal t_1 together with the order topology. Denote by L the set of ordinals in Ω with no immediate predecessor (0 is in L) and by F the set of ordinals in L of the form $t + \omega$, where $t \in L$ and ω is the first infinite ordinal. Define the function x_1 on Ω as follows: $x_1(t) = 0$ if $t \in F$, $x_1(t) = 1$ if $t \in L - F$, and $x_1(t+n) = 1/n$, $n = 1,2,\cdots$, if $t \in L$ with $t \neq t_1$. Then x_1 is upper semicontinuous on Ω and continuous at each point of $F \cup (\Omega - L)$. Let X denote the uniform closure of the linear space of all functions of the form $f + gx_1$, where $f, g \in C(\Omega)$ and $f = 0$ on $L - F$. Then X is an USC scalar function space on Ω since X is invariant under multiplication by elements of $C(\Omega)$ and for each $x \in X$, the function $|x|$ is upper semicontinuous on Ω. Thus X is a square space by Lemma 1. Further, $E = \{\pm \hat{t}: t \in \Omega\}$ by Lemma 2, since for each $t \in \Omega$ there exists $x \in X$ with $x(t) \neq 0$. Hence the map $t \mapsto \{\hat{t}, -\hat{t}\}$ is a bijection of Ω onto E_σ. The weak topology induced by this map, which we call the structure topology on Ω, has as base the family of sets of the form $V_x = \{t \in \Omega: x(t) \neq 0\}$, for $x \in X$. To prove that E_σ is not structurally regular, we will show that the point t_1 and the structurally closed set F (which is just $\Omega - V_{x_1}$) do not have disjoint structural neighborhoods. For this it suffices to show that if $x \in X$ with $x(t_1) \neq 0$, then there exists $t \in F$ with the property that $y \in X$ and $y(t) \neq 0$ imply $V_y \cap V_x \neq \varnothing$. So suppose $x(t_1) \neq 0$ and let $\alpha = x(t_1)$. We may assume (for simplicity) that $\alpha > 0$. By definition of X there exist sequences $\{f_n\}$ and $\{g_n\}$ in $C(\Omega)$ such that $f_n + g_n x_1$ converges uniformly to x on Ω and each $f_n = 0$ on $L - F$. Then $g_n(t_1) \to \alpha$ since t_1 is in $L - F$. Further, for each n there is $s_n < t_1$ such that $f_n = 0$ and $g_n = K_n$ (a constant) on the interval $(s_n, t_1]$. Let $s = \sup_n s_n$ and let $U = (s, t_1]$. Then, since $K_n = g_n(t_1) \to \alpha$, we have for $t \in U$ and n sufficiently large,

$$f_n(t) + g_n(t)x_1(t) = K_n x_1(t) \geqq \alpha x_1(t)/2.$$

Therefore $x(t) \geqq \alpha x_1(t)/2$ for all t in U. Choose t in $F \cap U$, and suppose $y \in X$ with

$y(t) \neq 0$. Since y is continuous at t (because $t \in F$) we have $y(t') \neq 0$ for some $t' < t$ with t' in $U - L$. Then $x_1(t') > 0$ and so

$$x(t') \geqq \alpha x_1(t')/2 > 0.$$

Therefore t' is in $V_y \cap V_x$.

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